

## Key

1. For all **natural numbers**  $n$ , let  $S(n)$  be the sum of the digits of  $n$  plus the number of digits of  $n$ . For instance,  $S(125) = 1 + 2 + 5 + 3 = 11$ . Note that the first digit of  $n$ , when reading from left to right, cannot be zero.
  - a. Determine  $S(12408)$
  - b. Determine all numbers  $m$  such that  $S(m) = 4$
  - c. Determine whether or not there exists a natural number  $m$  such that  $S(m) - S(m + 1) > 50$ . Provide a clear justification for your answer.

### Problem 1 Solution:

- a.  **$S(12408) = 1 + 2 + 4 + 0 + 8 + 5 = 20$ .**
- b. Consider 1-digit numbers, then  $m = 3$ . Among 2-digit numbers we need those with the sum of their digits equal to 2; so we have  $m = 11$  and  $m = 20$ . Among 3-digit numbers we need those with the sum of their digits equal to 1, so we have  $m = 100$ . For numbers with 4 or greater than 4 digits, we need those with the sum of their digits equal to 0, so we have  $m = 1000$ .

2. Suppose line J in the xy-

3. Suppose  $g(x)$  is the quadratic function  $g(x) = x^2 - ax + b$ , where  $a$  and  $b$  are **natural numbers**.
- If  $a = b = 2$ , find the set of real roots of the expression  $g(x) - x$ .
  - If  $a = b = 2$ , find the set of real roots of the expression  $g(g(x)) - x$ .
  - Find the number of pairs of **natural numbers**  $(a, b)$  such that  $a \leq 22$ ,  $b \leq 22$ , and every root of the expression  $g(g(x)) - x$  is an integer.

### Problem 3 Solution

- If  $a = 2$  and  $b = 2$ , then  $g(x) = x^2 - 2x + 2$ .  
Therefore, the roots of  $g(x) - x = x^2 - 3x + 2 = (x-1)(x-2) = 0$  are  $x = 1, 2$ .
- We now determine  $g(g(x)) - x$ . Note that  $g(g(x)) = (x^2 - 2x + 2)^2 - 2(x^2 - 2x + 2) + 2 = x^4 - 4x^3 + 8x^2 - 8x + 6$ .

Since  $1 \leq t \leq 22$ ,  $2 \leq 2t \leq 44$ ,  $2 \leq 2t-1 \leq 43$ ,  $2 \leq 2t+1 \leq 45$ . 2023  
 $= 44$ , where  $t$  denotes the largest integer less than or equal to  $t$ . There are 43 solutions

4. Jaden takes a mathematics test consisting of 100 questions, where the answer to each question is either TRUE or FALSE. For every five consecutive questions on the test, the answers to exactly three of the questions are TRUE. If the answers to Question 1 and Question 100 are both FALSE:

a.

5. Suppose quadrilateral STRV is an isosceles trapezoid, with  $ST = 5$  cm,  $RV = 5$  cm,  $TR = 2$  cm, and  $SV = 8$  cm.
- What is the length of diagonal SR?
  - For the isosceles trapezoid in part (a), what is the exact value of the **cosine** of  $\angle RTS$ ?
  - In triangle KLM below, points G and E are points on segment LM so that  $\angle MKG = \angle GKE = \angle EKL$ . Also, point F is located on segment KL so that segment GF is parallel to segment KM. If quadrilateral KFEG is an isosceles trapezoid and the measure of  $\angle KLM$  is  $84^\circ$ , find the measure of  $\angle MKG$ .

**Problem 5 Solution:**

- Let TE be the altitude of the trapezoid, so that  $\triangle TES$  is a right triangle with hypotenuse  $ST = 5$  and  $SE = \frac{8^2 - 2^2}{2} = 3$ .  
Therefore, the altitude is 4 (the sides of triangle TES form the Pythagorean triple 3-4-5).

Using the fact that the altitude of the trapezoid is 4, construct altitude SF from point S. **From the right triangle SFR, where  $SF = 4$ ,  $FR = FT + TR = 3 + 2 = 5$ , we find  $SR =$**

b. Using triangle STR and the Law of  
RTS.

So,  
 $\cos \text{RTS} = \frac{0}{F}$

As an alternative approach, since FT =  
SE = 3, and ST=5,  $\cos \text{FTS} = \frac{3}{5}$ .

Then  $\cos \text{RTS} = \frac{0}{F} + \cos \text{FTS} =$   
F

6. If  $M$  is a **natural number**